

THE LAX-OLEINIK SEMIGROUP ON GRAPHS

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ABSTRACT. We consider Tonelli Lagrangians on a graph, define weak KAM solutions, prove that they are the fixed points of the Lax-Oleinik semi-group and identify their uniqueness set as the Aubry set, giving a representation formula. Our main result is the long time convergence of the Lax Oleinik semi-group. We also prove that weak KAM solutions are viscosity solutions, and in the case of Hamiltonians of eikonal type, that viscosity solutions are weak KAM solutions

1. INTRODUCTION

We study the Lax-Oleinik semi-group defined by a Tonelli Lagrangian on a graph and prove its long time convergence. For Lagrangians on compact manifolds a standard proof uses the Euler-Lagrange flow and conservation of energy. In our case we do not have these tools but we can follow ideas of Roquejoffre [R] and Davini-Siconolfi [DS].

Camilli and collaborators [ACCT, CM, CS] have studied viscosity solutions of the Hamilton-Jacobi equation and given sufficient conditions for a set to be a uniqueness set and a representation formula.

In this paper we consider a graph G without boundary consisting of unoriented edges $\{I_j\}$ and vertices $\{v_a\}$. We will make the convention that at a vertex v_a the zero vectors in $T_{v_a}I_j$ all coincide. A Lagrangian in G is a C^k , strictly convex and super-linear function $L : TG \rightarrow \mathbb{R}$ given, according to our convention, by a collection of C^k functions $L_j : TI_j \rightarrow \mathbb{R}$ such that $L_j|_{T_x I_j}$ is convex and super-linear for any $x \in I_j$, any j , and $L_i(v_a, 0) = L_j(v_a, 0)$ for $v_a \in I_i \cap I_j$. We will say that a Lagrangian is *symmetric at the vertices* if $L_i(v_a, z) = l_a(|z|)$ for $v_a \in I_j$, $z \in T_{v_a}I_j$. As an example we have a mechanical Lagrangian. For x in the interior of I_j , it makes sense to say that $\zeta \in T_x I_j - \{0\}$ points towards one of the vertices of I_j , and for $v_a \in I_j$ it makes sense to say that $\zeta \in T_{v_a}I_j - \{0\}$ is incoming or outgoing. We set

$$\begin{aligned} T_{v_a}^+ I_j &= \{z \in T_{v_a} I_j : z \text{ is outgoing or } z = 0\}, \\ T_{v_a}^- I_j &= \{z \in T_{v_a} I_j : z \text{ is incoming or } z = 0\}. \end{aligned}$$

2. BASIC PROPERTIES OF THE ACTION

The purpose of this section is to obtain the lower semicontinuity of the action and apriori bounds for the $W^{1,\infty}$ norm of minimizers. Denote by $\mathcal{C}^{ac}([a, b])$ the set

of absolutely continuous functions $\gamma : [a, b] \rightarrow G$ provided with the topology of uniform convergence. For $\gamma \in \mathcal{C}^{ac}([a, b])$ we define its action by

$$A(\gamma) = \int_a^b L(\gamma(t)\dot{\gamma}(t))dt$$

A minimizer is a $\gamma \in \mathcal{C}^{ac}([a, b])$ such that for any $\alpha \in \mathcal{C}^{ac}([a, b])$ with $\alpha(a) = \gamma(a)$, $\alpha(b) = \gamma(b)$ we have

$$A(\gamma) \leq A(\alpha)$$

The following two properties of the Lagrangian are important to achieve our goal and follow from its convexity and super-linearity.

Proposition 1. *If $C \geq 0$, $\varepsilon > 0$, there is $\eta > 0$ such that for $x, y \in I_j$, $d(x, y) < \eta$ and $v \in T_x G$, $|v| \leq C$, $w \in T_y G$ we have*

$$L(y, w) \geq L(x, v) + L_v(x, v)(w - v) - \varepsilon.$$

Proposition 2. *If $L_{vv} \geq \theta > 0$, $C \geq 0$, $\varepsilon > 0$, there is $\eta > 0$ such that for $x, y \in I_j$, $d(x, y) < \eta$ and $v \in T_x G$, $|v| \leq C$, $w \in T_y G$ we have*

$$L(y, w) \geq L(x, v) + L_v(x, v)(w - v) + \frac{3\theta}{4}|w - v|^2 - \varepsilon.$$

Lemma 1. *Let L be a Lagrangian on G . If a sequence $\gamma_n \in \mathcal{C}^{ac}([a, b])$ converges uniformly to the curve $\gamma : [a, b] \rightarrow G$ and*

$$\liminf_{n \rightarrow \infty} A(\gamma_n) < \infty$$

then the curve γ is absolutely continuous and

$$\liminf_{n \rightarrow \infty} A(\gamma_n) \geq A(\gamma).$$

Proof. Let $l = \liminf_{n \rightarrow \infty} A(\gamma_n)$. Passing to a subsequence we can assume that

$$A(\gamma_n) < l + 1, \quad \forall n \in \mathbb{N}$$

By the superlinearity we may assume that $L \geq 0$. Fix $\varepsilon > 0$ and take $B > 2(l+1)/\varepsilon$. Again by superlinearity there is a positive number $C(B)$ such that

$$L(x, v) \geq B\|v\| - C(B), \quad x \in G, v \in T_x G$$

Since $L \geq 0$, for $E \subset [a, b]$ measurable we have

$$-C(B)\text{Leb}(E) + B \int_E \|\dot{\gamma}_n\| \leq \int_E L(\gamma_n, \dot{\gamma}_n) + \int_{[a, b] \setminus E} L(\gamma_n, \dot{\gamma}_n) \leq l + 1.$$

Thus

$$\int_E \|\dot{\gamma}_n\| \leq \frac{1}{B}(l + 1 + C(B)\text{Leb}(E)) \leq \frac{\varepsilon}{2} + \frac{C(B)\text{Leb}(E)}{B}.$$

Choosing $0 < \delta < \frac{\varepsilon B}{2C(B)}$ we have that

$$\text{Leb}(E) < \delta \Rightarrow \forall n \in \mathbb{N} \int_E \|\dot{\gamma}_n\| < \varepsilon.$$

Since the $\dot{\gamma}_n$ are uniformly integrable, we have that they converge to $\dot{\gamma}$ in the $\sigma(L^1, L^\infty)$ weak topology.

We may assume that $\gamma_n([a, b])$ is contained in a compact neighbourhood K of $\gamma([a, b])$.

Let $\varepsilon > 0$ and $E_C = \{t : |\dot{\gamma}(t)| \leq C, d(\gamma(t), \{v_a\}) \geq \frac{1}{C}\}$, then by Proposition 1, for n large,

$$(1) \quad \int_{E_C} [L(\gamma, \dot{\gamma}) + L_v(\gamma, \dot{\gamma})(\dot{\gamma}_n - \dot{\gamma}) - \varepsilon] \leq \int_{E_C} L(\gamma_n \dot{\gamma}_n)$$

Letting $n \rightarrow +\infty$ on (1), we have that

$$\int_{E_C} L(\gamma, \dot{\gamma}) - \varepsilon (b - a) \leq l.$$

Since $E_C \uparrow [a, b]$ when $C \rightarrow +\infty$ and $L \geq 0$, we have

$$A(\gamma) = \lim_{C \rightarrow +\infty} \int_{E_C} L(\gamma, \dot{\gamma}) \leq l + \varepsilon (b - a).$$

Now let $\varepsilon \rightarrow 0$. □

Lemma 1 imply

Theorem 1. *Let L be a Lagrangian on G . The action $A : \mathcal{C}^{ac}([a, b]) \rightarrow \mathbb{R} \cup \{\infty\}$ is lower semicontinuous.*

Proposition 3. *Let L be a Lagrangian on G .*

The set $\{\gamma \in \mathcal{C}^{ab}([a, b]) : A(\gamma) \leq K\}$ is compact with the topology of uniform convergence.

Let $C_t = \sup\{L(x, v) : x \in G, |v| \leq \frac{\text{diam}(G)}{t}\}$, then for any minimizer $\gamma : [a, b] \rightarrow G$ with $b - a \geq t$ we have

$$A(\gamma) \leq C(b - a).$$

Proposition 4. *Suppose $\gamma_i : [a, b] \rightarrow G$ converge uniformly to $\gamma : [a, b] \rightarrow G$ and $A(\gamma_i)$ converges to $A(\gamma)$, then $\dot{\gamma}_i$ converges to $\dot{\gamma}$ in $L^1[a, b]$*

Proof. From Lemma 1 we have that if $F \subset [a, b]$ is a finite union of intervals

$$\int_F L(\gamma, \dot{\gamma}) \leq \liminf_n \int_F L(\gamma, \dot{\gamma}) \text{ and } \int_{F^c} L(\gamma, \dot{\gamma}) \leq \liminf_n \int_{F^c} L(\gamma, \dot{\gamma})$$

Since

$$\lim_n \int_F L(\gamma_n, \dot{\gamma}_n) + \int_{F^c} L(\gamma_n, \dot{\gamma}_n) = \lim_n A(\gamma_n) = A(\gamma)$$

we have

$$(2) \quad \lim_n \int_{F^c} L(\gamma_n, \dot{\gamma}_n) = \int_{F^c} L(\gamma, \dot{\gamma})$$

For $l > 0$ let F be a finite union of intervals such that

$$D_l := \{t \in [a, b] : |\dot{\gamma}(t)| > l, d(\gamma(t), \{v_a\}) < \frac{1}{l}\} \subset F, \text{ and } \text{Leb}(F \setminus D_l) < \frac{1}{l}.$$

Given $\varepsilon > 0$, from Proposition 2 we have that for n large enough

$$(3) \quad \frac{3\theta}{4} \int_{F^c} |\dot{\gamma}_n - \dot{\gamma}|^2 \leq \int_{F^c} L(\gamma_n, \dot{\gamma}_n) - \int_{F^c} L(\gamma, \dot{\gamma}) - \int_{F^c} L_v(\gamma, \dot{\gamma})(\dot{\gamma}_n - \dot{\gamma}) + \varepsilon.$$

As in Lemma 1, the $\dot{\gamma}_n$ are uniformly integrable, so they converge to $\dot{\gamma}$ in the $\sigma(L_1, L_\infty)$ weak topology and then

$$(4) \quad \lim_n \int_{F^c} L_v(\gamma, \dot{\gamma})(\dot{\gamma}_n - \dot{\gamma}) = 0$$

From (2), (3), (4) we get that $\limsup_n \int_{F^c} |\dot{\gamma}_n - \dot{\gamma}|^2 \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary

$$(5) \quad \lim_n \int_{F^c} |\dot{\gamma}_n - \dot{\gamma}|^2 = 0.$$

Take $B > 0$ such that for any $x \in G$ and $v \in T_x M$ $L(x, v) \geq \frac{\theta}{2}|v|^2 - B$. Then

$$\begin{aligned} \int_F |\dot{\gamma}_n - \dot{\gamma}|^2 &\leq \int_F 2|\dot{\gamma}_n|^2 + 2|\dot{\gamma}|^2 \\ &\leq \frac{4}{\theta} \int_F L(\gamma_n, \dot{\gamma}_n) + \frac{4}{\theta} \int_F L(\gamma, \dot{\gamma}) + \frac{8B}{\theta} \text{Leb}(F). \end{aligned}$$

If l is large enough, $L(\gamma, \dot{\gamma}) > 0$ on F . By (2), for n big enough $\int_F L(\gamma_n, \dot{\gamma}_n) \leq$

$2 \int_F L(\gamma, \dot{\gamma})$. Therefore

$$\limsup_n \int_F |\dot{\gamma}_n - \dot{\gamma}|^2 \leq \frac{12}{\theta} \int_F L(\gamma, \dot{\gamma}) + \frac{8B}{\theta} \text{Leb}(F).$$

Letting $l \rightarrow \infty$

$$\lim_n \int_a^b |\dot{\gamma}_n - \dot{\gamma}|^2 = 0,$$

which implies by Cauchy-Schwartz inequality that

$$\lim_n \int_a^b |\dot{\gamma}_n - \dot{\gamma}| = 0.$$

□

Lemma 2. *Let L be a Lagrangian in G . For $\varepsilon > 0$ there exists K_ε that is a Lipschitz constant for any minimizer $\gamma : [a, b] \rightarrow G$ with $b - a \geq \varepsilon$.*

Proof. If γ is a minimizer and $\gamma((c, d)) \subset I_j$ then $\gamma|_{(c, d)}$ is a solution of the Euler Lagrange equation for L_j .

Suppose the Lemma is not true, then there are minimizers $\gamma_i : [a_i, b_i] \rightarrow G$ and $c_i \in [a_i, b_i]$ such that $\dot{\gamma}(c_i) \rightarrow \infty$ when $i \rightarrow \infty$. Translating in time, we can assume that $c_i = c$ for all i and taking a subsequence that there is $a \in \mathbb{R}$ such that γ_i is defined in $[a, a + \frac{\varepsilon}{2}]$, $c \in [a, a + \frac{\varepsilon}{2}]$. As $A(\gamma_i|_{[a, a + \frac{\varepsilon}{2}]})$ is bounded, there is subsequence $\gamma_i|_{[a, a + \frac{\varepsilon}{2}]}$ which converges uniformly to $\gamma : [a, a + \frac{\varepsilon}{2}] \rightarrow G$. Since γ is limit of minimizers, it is a minimizer and $A(\gamma) \leq \liminf A(\gamma_i|_{[a, a + \frac{\varepsilon}{2}]})$. We can not have that $A(\gamma) < \limsup A(\gamma_i|_{[a, a + \frac{\varepsilon}{2}]})$ because that would contradict that the γ_i are minimizers. Thus $A(\gamma) = \lim A(\gamma_i|_{[a, a + \frac{\varepsilon}{2}]})$.

If $\gamma(c)$ is within I_j , there is $\delta > 0$ such that $\gamma([c - \delta, c + \delta]) \subset I_j$.

If $\gamma(c) = v_a$ we have 2 possibilities (not mutually exclusive)

1. $\gamma_i(c) \neq v_a$ for a infinitely many i 's. We have 2 possibilities (not exclusive)

a) There is an edge I_j with $v_a \in I_j$ and infinitely many i 's such that $\gamma_i(c) \in I_j$ and $\dot{\gamma}_i(c)$ points towards v_a .

b) There is an edge I_j with $v_a \in I_j$ and infinitely many i 's such that $\gamma_i(c) \in I_j$ and $\dot{\gamma}_i(c)$ points towards the other vertex.

2. $\gamma_i(c) = v_a$ for infinitely many i 's. We have 2 possibilities (not exclusive)

a) There is an edge I_j with $v_a \in I_j$ and infinitely many i 's such that $\dot{\gamma}_i(c)$ is outgoing.

b) There is an edge I_j with $v_a \in I_j$ and infinitely many i 's such that $\dot{\gamma}_i(c)$ is incoming.

In cases a) there is $\delta > 0$ such that $\gamma([c - \delta, c]) \subset I_j$.

In cases b) there is $\delta > 0$ such that $\gamma([c, c + \delta]) \subset I_j$.

We have that γ is a solution of the Euler-Lagrange equation for L_j either on $[c - \delta, c]$ or on $[c, c + \delta]$ and then $|\dot{\gamma}(t)| \leq K$ on $[c - \delta, c]$ or $[c, c + \delta]$. For some $0 < \delta_1 < \delta$ we have that γ_i are solutions of the Euler-Lagrange equation for L_j on $[c - \delta_1, c]$ or on $[c, c + \delta_1]$. For i sufficiently large, we have that $|\dot{\gamma}_i(t)| > 2K$ either in $[c - \delta_1, c]$ or in $[c, c + \delta_1]$. This would contradict Proposition 4 \square

3. THE PEIERLS BARRIER

The content of this section is similar to that for the case of Lagrangians on compact manifolds. Given $x, y \in G$ let $\mathcal{C}^{ac}(x, y, T)$ be the set curves $\alpha \in \mathcal{C}^{ac}([0, T])$ such that $\alpha(0) = x$ and $\alpha(T) = y$. For a given real number k define

$$h_T(x, y) = \min_{\alpha \in \mathcal{C}^{ac}(x, y, T)} A(\alpha)$$

and

$$h^k(x, y) = \liminf_{T \rightarrow \infty} h_T(x, y) + kT$$

Lemma 3. *For $\varepsilon > 0$ the function $F : [\varepsilon, \infty) \times G \times G \rightarrow \mathbb{R}$ defined by $F(t, x, y) = h_t(x, y)$ is Lipschitz*

Proof. By Proposition 2 there is $K > 0$ that is a Lipschitz constant for any minimizer $\gamma : [0, t] \rightarrow G$ with $t \geq \varepsilon$, so that $\|\dot{\gamma}\| \leq K$ almost everywhere.

Let $\Delta = \max(1, \text{diam}(G))$ and

$$B = \max\{|L(x, v)| : \|v\| \leq K + 3\Delta\}.$$

Let $t \geq \varepsilon$, $x, y, x', y' \in M$ and $\delta = \min(\varepsilon/3, d(x, x'))$, $\delta' = \min(\varepsilon/3, d(y, y'))$. There is a minimizer $\gamma \in \mathcal{C}^{ac}(x, y, t)$ such that $A(\gamma) = h_t(x, y)$. Let $\beta : [0, \delta] \rightarrow G$ be a minimizing geodesic between x' and $\gamma(\delta)$ so that

$$\|\dot{\beta}\| \leq \frac{d(x', \gamma(\delta))}{\delta} \leq \frac{d(x, x') + d(x, \gamma(\delta))}{\delta} \leq \frac{d(x, x')}{\delta} + K \leq 3\Delta + K$$

and hence $A(\beta) \leq B\delta$. Similarly, for $\beta' : [t - \delta', t] \rightarrow G$ a minimizing geodesic between $\gamma(t - \delta')$ and y' we have $A(\beta') \leq B\delta'$. Thus

$$\begin{aligned} h_t(x', y') &\leq h_\delta(x', \gamma(\delta)) + h_{t-\delta+\delta'}(\gamma(\delta), \gamma(t - \delta')) + h_{\delta'}(\gamma(t - \delta'), y') \\ &\leq h_{t-\delta-\delta'}(\gamma(\delta), \gamma(t - \delta')) + B\delta + B\delta' \\ &\leq h_{a,b}(x, y) - A(\gamma|_{[0, \delta]}) - A(\gamma|_{[t-\delta', t]}) + B\delta + B\delta' \\ &\leq h_t(x, y) + 2Bd(x, x') + 2Bd(y, y') \end{aligned}$$

which proves that $2B$ is a Lipschitz constant for all functions h_t with $t \geq \varepsilon$.

Let $s, t \in \mathbb{R}$ such that $\varepsilon \leq t \leq s$ and $\gamma \in \mathcal{C}^{ac}(x, y, s)$ such that $A(\gamma) = h_s(x, y)$. Then

$$h_s(x, y) = h_t(x, \gamma(t)) + h_{s-t}(\gamma(t), y).$$

Since $\|\dot{\gamma}\| \leq K$ almost everywhere, the definition of B gives

$$h_{s-t}(\gamma(t), y) \leq B(s - t).$$

Since $2B$ is a Lipschitz constant for h_t

$$h_t(x, \gamma(t)) - h_t(x, y) \leq 2Bd(\gamma(t), y) \leq 2BK(s - t).$$

Then

$$h_s(x, y) - h_t(x, y) \leq B(2K + 1)(s - t).$$

□

Lemma 4. *There exists a real c independent of x and y such that*

- (1) *For all $k > c$ we have $h^k(x, y) = \infty$.*
- (2) *For all $k < c$ we have $h^k(x, y) = -\infty$*
- (3) *$h^c(x, y)$ is finite. The function $h := h^c$ is called the Peierls barrier.*

Proof. Fix $x, y \in G$. Let \mathbf{A} be the set of k such that $h^k(x, y) = -\infty$, and \mathbf{B} be the set of k such that $h^k(x, y) = \infty$.

As function of k , $h^k(x, y)$ is monotone increasing. So if $h^k(x, y) = \infty$ for some k then $h^{k'}(x, y) = \infty$ for all $k' \geq k$, and analogously, if $h^k(x, y) = -\infty$ for some k then $h^{k'}(x, y) = -\infty$ for all $k' \leq k$. The set \mathbf{B} is nonempty, since for k large enough we have $L + k \geq 1$ and then $h_T(x, y) + kT \geq T$. Analogously the sets \mathbf{A} is nonempty, since taking a closed curve $\gamma : [0, 1] \rightarrow G$ and $k < -A(\gamma)$, we only have to go around γ more and more times.

Let c be the supremum of \mathbf{A} , and we next show that it is also the infimum of \mathbf{B} . Let $\varepsilon > 0$ then $h^{c+\varepsilon}(x, y) > -\infty$, so it is either finite or $+\infty$. In the first case, for each T we have

$$h_T^{c+2\varepsilon}(x, y) = \liminf h_T(x, y) + (c + \varepsilon)T + \varepsilon T = \infty.$$

Since ε is arbitrary, it follows that c is the infimum of \mathbf{B} .

Given other points $x', y' \in G$, for $T \geq 3$ we have

$$h_T(x', y') \leq h_1(x', x) + h_{T-2}(x, y) + h_1(y, y')$$

Therefore $h^k(x', y')$ is $-\infty$ if $h^k(x, y) = -\infty$. Changing the roles we conclude that the break point does not depend on the points.

It remains to prove that at the break point the value is finite. We need the following two Lemmas

Lemma 5. *Let $f_n : [0, n] \rightarrow \mathbb{R}$ be a family of continuous functions such that $f_n(0) = 0, f_n(n) = a_n$ and $\lim_{n \rightarrow \infty} \frac{a_n}{n} = 0$ then for any $\varepsilon > 0$ and any T there exist an n and an interval $[a, b]$ contained in $[0, n]$ such that $b - a > T$ and $|f(b) - f(a)| \leq \varepsilon$.*

Lemma 6. *The value c is the infimum of k such that $\int_{\gamma} L + k \geq 0$ for all closed curves γ .*

We show first that $h^c(x, y) > -\infty$. Assume the opposite, then using a curve $\alpha : [0, 1] \rightarrow G$ with $\alpha(0) = y, \alpha(1) = x$ we get a closed curve γ with negative $\int_{\gamma} L + c$, a contradiction with Lemma 6.

To prove that $h^c(x, y) < \infty$, let a_n be the infimum of $\int_{\gamma} L + c$ over all closed curves γ with period less or equal to n . The sequence (a_n) is non negative by Lemma 6 and $\lim_{n \rightarrow \infty} \frac{a_n}{n} = 0$. Indeed, if $l := \limsup_{n \rightarrow \infty} \frac{a_n}{n} > 0$, there is a sequence $n_k \rightarrow \infty$ and closed curves γ_k with period less or equal to n_k such that $\int_{\gamma_k} L + c \geq \frac{ln_k}{2}$. Thus

$$\int_{\gamma_k} L + c - \frac{l}{4} \geq \frac{ln_k}{4} \text{ a contradiction with } c = \inf \mathbf{B}.$$

Let γ_n be a curve such that $|\int_{\gamma_n} L + c - a_n| < 1$, and $f_n(t) = \int_0^t L(\gamma_n(s), \dot{\gamma}_n(s))ds + ct$. The functions f_n satisfy the hypothesis of the Lemma 5 so given any T we can find a curve $\gamma : [0, T'] \rightarrow G$ with $T' > T$ joining some points x' and y' such that $A(\gamma) + cT' \leq \varepsilon$. Thus

$$h_{T'+2}(x, y) + c(T' + 2) \leq h_1(x, x') + \varepsilon + h_1(y', y) + 2c$$

is bounded independently of T . \square

Proof of Lemma 5. For simplicity we assume that the functions are monotone. Given $T > 0$ and $\varepsilon > 0$ let n be such that $n > Ta_n/\varepsilon$. This is possible due to the sub-linear growth of a_n . We divide the interval in a_n/ε pieces of length T . One of this pieces must have an increment less than ε otherwise the total increment would be larger than a_n , a contradiction. \square

Proof of Lemma 6. If $k \geq c$ and γ is a closed curve then $\int_{\gamma} L + k \geq 0$ otherwise we can, by going several times around the same curve, make the action arbitrarily negative. On the other hand if $k < c$ then $h^k(x, x) = -\infty$, so there are closed curves with negative action. \square

Definition 1. The *Mañé potencial* $\Phi : G \times G \rightarrow \mathbb{R}$ is defined by

$$\Phi(x, y) = \inf_{t>0} h_t(x, y) + ct.$$

Clearly we have $\Phi(x, y) \leq h(x, y)$ for any $x, y \in G$.

Proposition 5. *Functions h and Φ have the following properties.*

- (1) $\Phi(x, z) \leq \Phi(x, y) + \Phi(y, z)$.
 - (2) $h(x, z) \leq h(x, y) + \Phi(y, z)$, $h(x, z) \leq \Phi(x, y) + h(y, z)$.
 - (3) h and Φ are Lipschitz
 - (4) If $\gamma_n : [0, t_n] \rightarrow G$ is a sequence of absolutely continuous curves with $t_n \rightarrow \infty$ and $\gamma_n(0) \rightarrow x$, $\gamma_n(t_n) \rightarrow y$, then
- $$(6) \quad h(x, y) \leq \liminf_{n \rightarrow \infty} A(\gamma_n) + ct_n.$$

Proof. For $x, y, z \in G$, $s, t > 0$

$$(7) \quad h_{s+t}(x, z) + c(s + t) \leq h_s(x, y) + h_t(y, z) + c(s + t)$$

(1) Taking $\inf_{s>0}$ in (7) we get

$$\Phi(x, z) \leq \Phi(x, y) + h_t(y, z) + ct.$$

Taking now $\inf_{t>0}$ we have

$$\Phi(x, z) \leq \Phi(x, y) + \Phi(y, z).$$

(2) Taking $\liminf_{s \rightarrow \infty}$ in (7) we get

$$h(x, z) \leq h(x, y) + h_t(y, z) + ct.$$

Taking now $\inf_{t > 0}$ we have

$$h(x, z) \leq h(x, y) + \Phi(y, z).$$

(3) Set $B = \sup\{L(y, v) : y \in G, |v| \leq 1\}$. Let $x, y, w, z \in G$, and let $\gamma : [0, d(y, z)] \rightarrow G$ be a unit speed geodesic connecting y to z .

$$\Phi(y, z) \leq \int_0^{d(y, z)} (L(\gamma, \dot{\gamma}) + c) \leq (B + c)d(y, z).$$

Thus

$$h(x, z) \leq h(x, y) + (B + c)d(y, z),$$

and exchanging the roles of y and z

$$h(x, y) \leq h(x, z) + (B + c)d(y, z).$$

In the same way

$$|h(x, z) - h(w, z)| \leq (B + c)d(x, w).$$

Writting

$$h(x, y) - h(w, z) = h(x, y) - h(x, z) + h(x, z) - h(w, z),$$

we get

$$|h(x, y) - h(w, z)| \leq (B + c)(d(y, z) + d(x, w)).$$

Similarly Φ is Lipschitz.

(4) Let K be a Lipschitz constant for all $h_t : t \geq 1$, then

$$\begin{aligned} h_{t_n}(x, y) &\leq h_{t_n}(\gamma_n(0), \gamma_n(0)) + K(d(x, \gamma_n(0)) + d(\gamma_n(t_n), y)) \\ &\leq A(\gamma_n) + K(d(x, \gamma_n(0)) + d(\gamma_n(t_n), y)). \end{aligned}$$

Thus

$$h(x, y) \leq \liminf_{n \rightarrow \infty} h_{t_n}(x, y) + ct_n \leq \liminf_{n \rightarrow \infty} A(\gamma_n) + ct_n.$$

□

Definition 2. A curve $\gamma : J \rightarrow G$ is called

- *semi-static* if

$$\Phi(\gamma(t), \gamma(s)) = \int_t^s L(\gamma, \dot{\gamma})$$

for any $t, s \in J, t \leq s$.

- *static* if

$$\int_t^s L(\gamma, \dot{\gamma}) = -\Phi(\gamma(s), \gamma(t))$$

for any $t, s \in J, t \leq s$.

The *Aubry set* \mathcal{A} is the set of points $x \in G$ such that $h(x, x) = 0$.

Notice that by item (2) in Proposition 5, $h(x, z) = \Phi(x, z)$ if $x \in \mathcal{A}$ or $z \in \mathcal{A}$.

Proposition 6. *If $\eta : \mathbb{R} \rightarrow G$ is static then $\eta(s) \in \mathcal{A}$ for any $s \in \mathbb{R}$.*

Proof. We define the ω -limit of η as

$$\omega(\eta) = \{x \in G : \exists t_n \rightarrow \infty \text{ such that } x = \lim_{n \rightarrow \infty} \eta(t_n)\}$$

Let $x \in \omega(\eta)$, then

$$\begin{aligned} 0 \leq h(\eta(s), \eta(s)) &\leq h(\eta(s), x) + \Phi(x, \eta(s)) \\ &\leq \lim_{n \rightarrow \infty} A(\eta|_{[s, t_n]}) + ct_n + \Phi(x, \eta(s)) \\ &\leq \lim_{n \rightarrow \infty} -\Phi(\gamma(t_n), \gamma(s)) + \Phi(x, \eta(s)) = 0 \end{aligned}$$

□

4. WEAK KAM SOLUTIONS

Following Fathi, we define weak KAM solutions and prove some of their properties

Definition 3. Let c be given by Lemma 4.

- A function $u : G \rightarrow \mathbb{R}$ is *dominated* if for any $x, y \in G$, we have

$$u(y) - u(x) \leq h_t(x, y) + ct \quad \forall t > 0,$$

or equivalently

$$u(y) - u(x) \leq \Phi(x, y).$$

- $\gamma : I \rightarrow G$ *calibrates* a dominated function $u : G \rightarrow \mathbb{R}$ if

$$u(\gamma(s)) - u(\gamma(t)) = \int_t^s L(\gamma, \dot{\gamma}) + c(s - t) \quad \forall s, t \in I$$

- A continuous function $u : G \rightarrow \mathbb{R}$ is a *backward (forward) weak KAM solution* if it is dominated and for any $x \in G$ there is $\gamma : (-\infty, 0] \rightarrow G$ ($\gamma : [0, \infty) \rightarrow G$) that calibrates u and $\gamma(0) = x$

Proposition 7. *For any $x \in G$, $h(x, \cdot)$ is a backward weak KAM solution and $-h(\cdot, x)$ is a forward weak KAM solution.*

Proof. Let $\gamma_n : [-t_n, 0] \rightarrow G$ be a sequence of minimizing curves connecting x to y such that

$$h(x, y) = \lim_{n \rightarrow \infty} A(\gamma_n) + ct_n$$

By Lemma 2, $\{\gamma_n\}$ is uniformly Lipschitz and then equicontinuous. For $l \in \mathbb{N}$ fixed, it follows from the Arzela Ascoli Theorem that there is a sequence $n_j \rightarrow \infty$ such that γ_{n_j} converges uniformly on $[-l, 0]$. Using a diagonal trick one obtains a sequence

$m_k \rightarrow \infty$ and $\gamma : (-\infty, 0] \rightarrow G$ such that γ_{m_k} converges to γ , uniformly on each $[-l, 0]$. Fix $t < 0$, for k large $t + m_k \geq 0$ and

$$(8) \quad A(\gamma_{m_k}) + cm_k = \int_{-t_{m_k}}^t L(\gamma_{m_k}, \dot{\gamma}_{m_k}) + c(t + m_k) + \int_t^0 L(\gamma_{m_k}, \dot{\gamma}_{m_k}) - ct.$$

Since γ_{m_k} converges to γ uniformly on $[t, 0]$, we have

$$\int_t^0 L(\gamma_{m_k}, \dot{\gamma}_{m_k}) \rightarrow \int_t^0 L(\gamma, \dot{\gamma}).$$

From item (4) of Proposition 5 we have

$$h(x, \gamma(t)) \leq \lim_{k \rightarrow \infty} \int_{-t_{m_k}}^t L(\gamma_{m_k}, \dot{\gamma}_{m_k}) + c(t + m_k).$$

Taking $\liminf_{k \rightarrow \infty}$ in (8) we get

$$h(x, y) \geq h(x, \gamma(t)) + \int_t^0 L(\gamma, \dot{\gamma}) - ct.$$

□

From Proposition 7 we have

Corollary 1. *If $x \in \mathcal{A}$ there exists a curve $\gamma : \mathbb{R} \rightarrow G$ such that $\gamma(0) = x$ and for all $t \geq 0$*

$$\begin{aligned} h(\gamma(t), x) &= - \int_0^t L(\gamma, \dot{\gamma}) - ct \\ h(x, \gamma(-t)) &= - \int_{-t}^0 L(\gamma, \dot{\gamma}) - ct. \end{aligned}$$

In particular the curve γ is static and calibrates any dominated function $u : G \rightarrow \mathbb{R}$.

Theorem 2. *The function $\Phi(x, \cdot)$ is a backward weak KAM solution if and only if $x \in \mathcal{A}$.*

Proof. If $x \in \mathcal{A}$ then $\Phi(x, \cdot) = h(x, \cdot)$ and so $\Phi(x, \cdot)$ is a backward weak KAM solution. Conversely if $\Phi(x, \cdot)$ is a backward weak KAM solution, let $\gamma : (-\infty, 0] \rightarrow G$ be such that $\gamma(0) = x$ and for all $t > 0$

$$\Phi(x, x) - \Phi(x, \gamma(-t)) = \int_{-t}^0 L(\gamma, \dot{\gamma}) + ct,$$

so that for all $t > 0$

$$\Phi(x, \gamma(-t)) + \int_{-t}^0 L(\gamma, \dot{\gamma}) + ct = 0.$$

For fixed $\varepsilon, T > 0$ let $\gamma_{\varepsilon, T} : [0, t_{\varepsilon, T}] \rightarrow G$ such that $\gamma_{\varepsilon, T}(0) = x$, $\gamma_{\varepsilon, T}(t_{\varepsilon, T}) = \gamma(-T)$ and

$$\int_0^{t_{\varepsilon, T}} L(\gamma_{\varepsilon, T}, \dot{\gamma}_{\varepsilon, T}) + ct \leq \Phi(x, \gamma(-T)) + \varepsilon.$$

Joining $\gamma_{\varepsilon, T}$ and $\gamma|_{[-T, 0]}$ we get a closed curve $\bar{\gamma}_{\varepsilon, T}$ such that

$$\int_0^{t_{\varepsilon, T} + T} L(\bar{\gamma}_{\varepsilon, T}, \dot{\bar{\gamma}}_{\varepsilon, T}) + c(t_{\varepsilon, T} + T) \leq \Phi(x, \gamma(-T)) + \varepsilon + \int_{-T}^0 L(\gamma, \dot{\gamma}) + cT \leq \varepsilon.$$

Since $\varepsilon, T > 0$ are arbitrary we have that $h(x, x) = 0$, which means $x \in \mathcal{A}$. \square

Corollary 2. *Let $C \subset G$ and $w_0 : C \rightarrow \mathbb{R}$ be bounded from below. Let*

$$w(x) = \inf_{z \in C} w_0(z) + \Phi(z, x)$$

- (1) *w is the maximal dominated function not exceeding w_0 on C .*
- (2) *If $C \subset \mathcal{A}$, w is a backward weak KAM solution.*
- (3) *If for all $x, y \in C$*

$$w_0(y) - w_0(x) \leq \Phi(x, y),$$

then w coincides with w_0 on C .

Proof. (1) That w is dominated follows from item (1) of Proposition 5. If φ is a dominated function with $\varphi \leq w_0$ on C ,

$$\varphi(x) \leq \min_{z \in C} \varphi(z) + \Phi(z, x) \leq w(x).$$

(2) if $C \subset \mathcal{A}$, for any $z \in C$ the function $\Phi(z, \cdot) = h(z, \cdot)$ is a backward weak KAM solution and so is w .

(3) is clear \square

For $u : G \rightarrow \mathbb{R}$ let $I(u)$ be the set of points $x \in G$ for which exists $\gamma : \mathbb{R} \rightarrow G$ such that $\gamma(0) = x$ and γ calibrates u .

Corollary 3.

$$\mathcal{A} = \bigcap_{u \text{ dominated}} I(u)$$

Proof. By corollary 1, \mathcal{A} is contained in the intersection on the r.h.s. If x belongs to the intersection, there is a curve $\gamma : \mathbb{R} \rightarrow G$ such that $\gamma(0) = x$ and γ calibrates $\Phi(x, \cdot)$ and we get as in the the proof of Theorem 2 that $x \in \mathcal{A}$. \square

Proposition 8. *For each $x, y \in G$ with $x \neq y$ we can find $\varepsilon > 0$ and a curve $\gamma : [-\varepsilon, 0] \rightarrow G$ such that $\gamma(0) = y$ and for all $t \in [0, \varepsilon]$*

$$\Phi(x, \gamma(0)) - \Phi(x, \gamma(-t)) = \int_{-t}^0 L(\gamma, \dot{\gamma}) + ct.$$

In particular, for each $x \in G$ the function $G \setminus \{x\} \rightarrow \mathbb{R}; y \mapsto \Phi(x, y)$ is a backward weak KAM solution.

Proof. We only need to consider the case when $x \notin \mathcal{A}$. In such a case we can find a sequence $t_n \rightarrow \bar{t} < \infty$ such that $h_{t_n}(x, y) + ct_n \rightarrow \Phi(x, y)$. Except for the first few terms we have $h_{t_n}(x, y) + ct_n \leq \Phi(x, y) + 1$. We can find curves $\gamma_n : [-t_n, 0] \rightarrow G$ with $\gamma_n(-t_n) = y$, $\gamma_n(0) = x$ and $A(\gamma_n) = h_{t_n}$. \square

Theorem 3. \mathcal{A} is nonempty and if $u : G \rightarrow \mathbb{R}$ is a backward weak KAM solution then

$$(9) \quad u(x) = \min_{q \in \mathcal{A}} u(q) + h(q, x)$$

Proof. Let $\gamma : (-\infty, 0] \rightarrow G$ be such that $\gamma(0) = x$ and for any $t < 0$

$$u(x) - u(\gamma(t)) = \int_t^0 L(\gamma, \dot{\gamma}) - ct.$$

For $s < t < 0$ we have

$$u(\gamma(t)) - u(\gamma(s)) = \int_s^t L(\gamma, \dot{\gamma}) + c(t - s) = h_{t-s}(\gamma(s), \gamma(t)) + c(t - s)$$

Choose a sequence $t_n \rightarrow -\infty$ such that $\gamma(t_n)$ converges to $p \in G$. By item 6 of Proposition 5

$$u(x) - u(p) = \lim_{n \rightarrow \infty} \int_{t_n}^0 L(\gamma, \dot{\gamma}) - ct_n \geq h(x, p).$$

We claim that $p \in \mathcal{A}$. Let $\delta_n = d(\gamma(t_n), p)$ and define B as in Proposition 5, then we have

$$h_{\delta_n}(\gamma(t_n), p), h_{\delta_n}(p, \gamma(t_n)) \leq B\delta_n.$$

Choose a sequence n_k such that $s_k = t_{n_k} - t_{n_{k+1}} \rightarrow \infty$ as $k \rightarrow \infty$. Putting $T_k = \delta_{n_{k+1}} + s_k + \delta_{n_k}$ we have

$$\begin{aligned} h_{T_k}(p, p) + cT_k &\leq h_{\delta_{n_{k+1}}}(p, \gamma(t_{n_{k+1}})) + h_{s_k}(\gamma(t_{n_{k+1}}), \gamma(t_{n_k})) + h_{\delta_{n_k}}(\gamma(t_{n_k}), p) + cT_k \\ &\leq (B + c)(\delta_{n_{k+1}} + \delta_{n_k}) + u(\gamma(t_{n_k})) - u(\gamma(t_{n_{k+1}})). \end{aligned}$$

Thus $h(p, p) \leq 0$ and $p \in \mathcal{A}$. Since u is dominated, for any $x \in G$

$$u(x) \leq \min_{q \in \mathcal{A}} u(q) + h(q, x)$$

\square

Corollary 4.

$$h(x, y) = \min_{q \in \mathcal{A}} h(x, q) + h(q, y) = \min_{q \in \mathcal{A}} \Phi(x, q) + \Phi(q, y)$$

5. THE LAX SEMIGROUP AND ITS CONVERGENCE

5.1. The Lax semigroup. Let \mathcal{F} be the set of bounded from below real function defined on G . The backward Lax semigroup $\mathcal{L}_t : \mathcal{F} \rightarrow \mathcal{F}$, $t > 0$ is defined by

$$\mathcal{L}_t f(x) = \inf_{y \in G} f(y) + h_t(y, x).$$

It is clear that $f \in \mathcal{F}$ is dominated if and only if $f \leq \mathcal{L}_t f + ct$ for any $t > 0$

Lemma 7. *Given $\varepsilon > 0$ there is $K_\varepsilon > 0$ such that for each $u : G \rightarrow \mathbb{R}$ continuous, $t \geq 0$, we have $\mathcal{L}_t u : G \rightarrow \mathbb{R}$ is a Lipschitz with constant K_ε .*

Proof. Let $K_\varepsilon > 0$ be a Lipschitz constant for the function $F : [\varepsilon, \infty) \times G \times G \rightarrow \mathbb{R}$ defined by $F(t, x, y) = h_t(x, y)$. Let $u : G \rightarrow \mathbb{R}$ be continuous and $x, y \in G$. Chose $z \in G$ such that $\mathcal{L}_t u(y) = u(z) + h_t(z, y)$. then

$$\mathcal{L}_t u(x) \leq u(z) + h_t(z, x) \leq u(z) + h_t(z, y) + K_\varepsilon d(x, y) = c\mathcal{L}_t u(y) + K_\varepsilon d(x, y).$$

Changing the roles of x and y we get that K_ε is a Lipschitz constant for $\mathcal{L}_t u$. \square

Theorem 4. *A continuous function $u : G \rightarrow \mathbb{R}$ is a fixed point of the semigroup $\mathcal{L}_t + ct$ if and only if it is a backward weak KAM solution*

Proof. Suppose $u : G \rightarrow \mathbb{R}$ is a fixed point of the semigroup $\mathcal{L}_t + ct$. For each $T \geq 2$ there is a curve $\alpha_T : [-T, 0] \rightarrow G$ such that $\alpha_T(0) = x$ and

$$u(x) - u(\alpha_T(-T)) = A(\alpha_T) + cT.$$

By Lemma 2 $\{\alpha_T\}$ is uniformly Lipschitz. As in Propostion 7 one obtains a sequence $t_k \rightarrow \infty$ and $\gamma : (-\infty, 0] \rightarrow G$ such that α_{t_k} converges to γ , uniformly on each $[-n, 0]$.

By Lemma 1

$$\begin{aligned} \int_{-n}^0 L(\gamma, \dot{\gamma}) + nc &\leq \liminf_{k \rightarrow \infty} \int_{-n}^0 L(\alpha_{t_k}, \dot{\alpha}_{t_k}) + nc \\ &= \liminf_{k \rightarrow \infty} u(x) - u(\alpha_{t_k}(-n)) \\ &= u(x) - u(\gamma(-n)) \end{aligned}$$

Suppose now that $u : G \rightarrow \mathbb{R}$ is a backward weak KAM solution. Since u is dominated, $u \leq \mathcal{L}_t u + ct$. For $x \in G$ let $\gamma : (-\infty, 0] \rightarrow G$ be such that $\gamma(0) = x$ and for all $t > 0$

$$u(x) - u(\gamma(-t)) = \int_{-t}^0 L(\gamma, \dot{\gamma}) + ct.$$

Thus

$$u(x) \geq u(\gamma(-t)) + h_t(\gamma(t), x) + ct \geq \mathcal{L}_t u(x) + ct.$$

\square

From Proposition 7 and Theorem 4 one obtains

Corollary 5. *The semigroup $\mathcal{L}_t + ct$ has Lipschitz fixed points*

5.2. Convergence of the Lax semigroup. Without loss of generality we assume $c = 0$. Given $u \in C(G)$, it is not hard to see that if $\mathcal{L}_t u$ converges as $t \rightarrow \infty$, then the limit must be

$$(10) \quad v(x) := \min_{z \in G} u(z) + h(z, x).$$

Using Corollary 4 we can write this function as

$$(11) \quad v(x) = \min_{y \in \mathcal{A}} \Phi(y, x) + w(y)$$

where

$$(12) \quad w(y) := \inf_{z \in G} u(z) + \Phi(z, y)$$

Item (1) of Corollary 2 states that w is the maximal dominated function not exceeding u . Items (2), (3) of the same Corollary imply that v is the unique backward weak KAM solution that coincides with w on \mathcal{A} .

Proposition 9. *Suppose that u is dominated, then $\mathcal{L}_t u$ converges uniformly as $t \rightarrow \infty$ to the function v given by (10)*

Proof. Since u is dominated, the function $t \mapsto \mathcal{L}_t u$ is nondecreasing. As well, in this case, w given by (12) coincides with u . Items (1) and (3) of Corollary 2 imply that v is the maximal dominated function that coincides with u on \mathcal{A} and then $u \leq v$ on G .

Since the semigroup \mathcal{L}_t is monotone and v is a backward weak KAM solution

$$\mathcal{L}_t u \leq \mathcal{L}_t v = v \text{ for any } t > 0.$$

Thus the uniform limit $\lim_{t \rightarrow \infty} \mathcal{L}_t u$ exists. □

For $u \in C(G)$ let

$$\omega_{\mathcal{L}}(u) := \{\psi \in C(G) : \exists t_n \rightarrow \infty \text{ such that } \psi = \lim_{n \rightarrow \infty} \mathcal{L}_{t_n} u\}.$$

$$(13) \quad \underline{u}(x) := \sup\{\psi(x) : \psi \in \omega_{\mathcal{L}}(u)\}$$

$$(14) \quad \bar{u}(x) := \inf\{\psi(x) : \psi \in \omega_{\mathcal{L}}(u)\}$$

Proposition 10. *For $u \in C(G)$, function \underline{u} given by (13) is dominated.*

Proof. Let $x, y \in G$. Given $\varepsilon > 0$ there is $\psi = \lim_{n \rightarrow \infty} \mathcal{L}_{t_n} u$ such that $\underline{u}(x) - \varepsilon < \psi(x)$.

For $n > N(\varepsilon)$ and $a > 0$

$$\underline{u}(x) - 2\varepsilon < \psi(x) - \varepsilon \leq \mathcal{L}_{t_n} u(x) = \mathcal{L}_a(\mathcal{L}_{t_n-a} u)(x) \leq \mathcal{L}_{t_n-a} u(y) + h_a(y, x).$$

Choose a divergent sequence n_j such that $(\mathcal{L}_{t_{n_j}-a} u)_j$ converges uniformly. For $j > \bar{N}(\varepsilon)$, $\mathcal{L}_{t_{n_j}-a} u(y) < \underline{u}(y) + \varepsilon$, and then

$$\underline{u}(x) - 3\varepsilon < \mathcal{L}_{t_{n_j}-a} u(y) + h_a(y, x) - \varepsilon < \underline{u}(y) + h_a(y, x).$$

□

Proposition 11. *Let $u \in C(G)$, v be the function given by (10), \underline{u}, \bar{u} defined in (13) and (14). Then*

$$(15) \quad v \leq \bar{u} \leq \underline{u}$$

Proof. Let w be given by (12). Since $w \leq u$, from the monotonicity of \mathcal{L}_t we get $\mathcal{L}_t w \leq \mathcal{L}_t u$. Since w is dominated, from Proposition 9 we have that $\mathcal{L}_t w$ converges to v and then $v \leq \bar{u}$. \square

Other proof. Let $\psi = \lim_{n \rightarrow \infty} \mathcal{L}_{t_n} u$. For $x \in G$ let $\gamma_n : [0, t_n] \rightarrow G$ be such that $\gamma_n(t_n) = x$ and

$$(16) \quad \mathcal{L}_{t_n} u(x) = u(\gamma_n(0)) + A(\gamma_n).$$

Passing to a subsequence if necessary we may assume that $\gamma_n(0)$ converges to $y \in G$. Taking \liminf in (16), we have from item (4) of Proposition 5

$$\psi(x) = u(y) + \liminf_{n \rightarrow \infty} A(\gamma_n) \geq u(y) + h(y, x) \geq v(x).$$

\square

Denote by \mathcal{K} the family of maximal static curves $\eta : \mathbb{R} \rightarrow G$, and for $y \in \mathcal{A}$ denote by $\mathcal{K}(y)$ the set of curves $\eta \in \mathcal{K}$ with $\eta(0) = y$.

Proposition 12. *\mathcal{K} is a compact metric space with respect to the uniform convergence on compact intervals.*

Proof. Let $\{\eta_n\}$ be a sequence in \mathcal{K} . By Lemma 2, $\{\eta_n\}$ is uniformly Lipschitz. As in Proposition 7 we obtain a sequence $n_k \rightarrow \infty$ such that η_{n_k} converges to $\eta : \mathbb{R} \rightarrow G$ uniformly on each $[a, b]$ and then η is static. \square

Proposition 13. *Let $\eta \in \mathcal{K}$. There exists a set $Z \subset \mathbb{R}$ with zero Lebesgue measure such that the functions η , $\Phi(\eta(t_0), \eta(\cdot))$ and $-\Phi(\eta(\cdot), \eta(t_0))$ are differentiable at any $t_0 \in \mathbb{R} \setminus Z$ and*

$$\Phi(\eta(t_0), \eta(\cdot))'(t_0) = -\Phi(\eta(\cdot), \eta(t_0))'(t_0) = L(\eta(t_0), \dot{\eta}(t_0))$$

Proof. By Rademacher and Lebesgue differentiability theorems, we can choose $Z \subset \mathbb{R}$ with zero Lebesgue measure such that every $t \in \mathbb{R} \setminus Z$ is a differentiability point for η and a Lebesgue point for the function $L(\eta(\cdot), \dot{\eta}(\cdot))$. As the curve η is static, for every $t > t_0$ we have

$$\frac{\Phi(\eta(t_0), \eta(t))}{t - t_0} = \frac{1}{t - t_0} \int_{t_0}^t L(\eta(s), \dot{\eta}(s)) ds.$$

For $t_0 \in \mathbb{R} \setminus Z$ we derive

$$\lim_{t \rightarrow t_0^+} \Phi(\eta(t_0), \eta(t)) = L(\eta(t_0), \dot{\eta}(t_0)).$$

Analogously we deduce a similar limit relation for $t \rightarrow t_0^-$. \square

Theorem 5. *Let $\eta \in \mathcal{K}$. Then there exists a set $Z \subset \mathbb{R}$ with zero Lebesgue measure such that for any dominated function φ , $\varphi \circ \eta$ is differentiable on $\mathbb{R} \setminus Z$ and*

$$(17) \quad (\varphi \circ \eta)'(t) = L(\eta(t), \dot{\eta}(t)) \quad \forall t \in \mathbb{R} \setminus Z$$

Thus all dominated functions coincide on $\eta(\mathbb{R})$, up to an additive constant.

Proof. Let Z and φ be the subset of \mathbb{R} given by Proposition 13 and a dominated function, respectively. For every $t, t_0 \in \mathbb{R}$,

$$-\Phi(\eta(t), \eta(t_0)) \leq \varphi(\eta(t)) - \varphi(\eta(t_0)) \leq \Phi(\eta(t_0), \eta(t))$$

and hence we get (17), for $t_0 \in \mathbb{R} \setminus Z$, in view of Proposition 13. \square

Proposition 14. *Let $\eta \in \mathcal{K}$, $\psi \in C(G)$ and φ be a dominated function. Then the function $t \mapsto (\mathcal{L}_t \psi)(\eta(t)) - \varphi(\eta(t))$ is nonincreasing on \mathbb{R}_+ .*

Proof. From Theorem 5, for $t < s$ we have

$$(\mathcal{L}_s \psi)(\eta(s)) - (\mathcal{L}_t \psi)(\eta(t)) \leq \int_t^s L(\eta(\tau), \dot{\eta}(\tau)) d\tau = \varphi(\eta(s)) - \varphi(\eta(t))$$

\square

Proposition 15. *Two dominated functions that coincide on $\mathcal{M} = \bigcup_{\eta \in \mathcal{K}} \omega(\eta)$ also coincide on \mathcal{A} .*

Proof. Let φ_1, φ_2 be two dominated functions coinciding on \mathcal{M} . Let $y \in \mathcal{A}$ and $\eta \in \mathcal{K}(y)$. Let $(t_n)_n$ be a diverging sequence such that $\lim_n \eta(t_n) = x \in \mathcal{M}$. As $\Phi(y, \cdot)$ is dominated Theorem 5 yields

$$\varphi_i(y) = \varphi_i(\eta(0)) - \Phi(y, \eta(0)) = \varphi_i(\eta(t_n)) - \Phi(y, \eta(t_n))$$

for every $n \in N, i = 1, 2$. Sending n to ∞ , we get

$$\begin{aligned} \varphi_1(y) &= \lim_{n \rightarrow \infty} \varphi_1(\eta(t_n)) - \Phi(y, \eta(t_n)) = \varphi_1(x) - \Phi(y, x) = \varphi_2(x) - \Phi(y, x) \\ &= \lim_{n \rightarrow \infty} \varphi_2(\eta(t_n)) - \Phi(y, \eta(t_n)) = \varphi_2(y) \end{aligned}$$

with y an arbitrary point of \mathcal{A} . \square

Proposition 16. *Let $\eta \in \mathcal{K}$. For almost every $t \in \mathbb{R}$*

$$L_v(\eta(t), \dot{\eta}(t)) \dot{\eta}(t) = L(\eta(t), \dot{\eta}(t))$$

Proof. For $\lambda > 0$, let $\eta_\lambda(t) := \eta(\lambda t)$ so that $\eta_\lambda(T/\lambda) = \eta(T)$ for any T , and $\dot{\eta}_\lambda(t) = \lambda \dot{\eta}(\lambda t)$ when $\lambda t \in \mathbb{R} \setminus Z$. For $T > 0$ let

$$\mathcal{A}_T(\lambda) := \int_0^{\frac{T}{\lambda}} L(\eta_\lambda(t), \dot{\eta}_\lambda(t)) dt = \int_0^T L(\eta(s), \lambda \dot{\eta}(s)) \frac{ds}{\lambda}.$$

Since η is a free-time minimizer, differentiating $\mathcal{A}_T(\lambda)$ at $\lambda = 1$, we have that

$$0 = \mathcal{A}'_T(1) = \int_0^T [L_v(\eta(s), \dot{\eta}(s))\dot{\eta}(s) - L(\dot{\eta}(s), \dot{\eta}(s))] ds.$$

Since this holds for any $T > 0$ we have

$$L_v(\eta(t), \dot{\eta}(t))\dot{\eta}(t) = L(\dot{\eta}(t), \dot{\eta}(t))$$

for almost every $t > 0$. Similarly for $t < 0$. \square

Lemma 8. *There is a $M > 0$ such that, if η is any curve in \mathcal{K} and λ is sufficiently close to 1, we have*

$$(18) \quad \int_{t_1}^{t_2} L(\eta_\lambda, \dot{\eta}_\lambda) \leq \Phi(\eta_\lambda(t_1), \eta_\lambda(t_2)) + M(t_2 - t_1)(\lambda - 1)^2$$

for any $t_2 > t_1$, where $\eta_\lambda(t) = \eta(\lambda t)$.

Proof. Let $K > 0$ be such that for any minimizer $\gamma : [a, b] \rightarrow G$ with $b - a > 1$, $|\dot{\gamma}(t)| < K$ and let $2R = \sup\{|L_{vv}(x, v)| : |v| \leq K\}$. Fix $\lambda \in (1 - \delta, 1 + \delta)$

$$\begin{aligned} \int_{t_1}^{t_2} L(\eta_\lambda(t), \dot{\eta}_\lambda(t)) dt &= \int_{t_1}^{t_2} [L(\eta(\lambda t), \dot{\eta}(\lambda t)) + (\lambda - 1)L_v(\eta(\lambda t), \dot{\eta}(\lambda t))\dot{\eta}(\lambda t) \\ &\quad + \frac{1}{2}(\lambda - 1)^2 L_{vv}(\eta(\lambda t), \mu \dot{\eta}(\lambda t))(\dot{\eta}(\lambda t))^2] dt \\ &\leq \lambda \int_{t_1}^{t_2} L(\eta(\lambda t), \dot{\eta}(\lambda t)) dt + (t_2 - t_1)RK^2(\lambda - 1)^2 \\ &= \Phi(\eta(\lambda t_1), \eta(\lambda t_2)) + (t_2 - t_1)RK^2(\lambda - 1)^2 \end{aligned}$$

\square

Proposition 17. *Let $\eta \in \mathcal{K}$, $\psi \in C(G)$ and φ be a dominated function. Assume that $D^+((\psi - \varphi) \circ \eta)(0) \setminus \{0\} \neq \emptyset$, where D^+ denotes the superdifferential, for all $t > 0$ we have*

$$(19) \quad (\mathcal{L}_t \psi)(\eta(t)) - \varphi(\eta(t)) < \psi(\eta(0)) - \varphi(\eta(0))$$

Proof. Fix $t > 0$. By Proposition 13 it is enough to prove (19) for $\varphi = -\Phi(\cdot, \eta(t))$. Since $\mathcal{L}_t(\psi + a) = \mathcal{L}_t \psi + a$ we can assume that $\psi(\eta(0)) = \varphi(\eta(0))$.

$$(\mathcal{L}_t \psi)(\eta(t)) - \varphi(\eta(t)) = (\mathcal{L}_t \psi)(\eta(t)) \leq \int_{(1/\lambda-1)t}^{t/\lambda} L(\eta_\lambda, \dot{\eta}_\lambda) + \psi(\eta((1 - \lambda)t)),$$

thus, by Lemma 8

$$(\mathcal{L}_t \psi)(\eta(t)) - \varphi(\eta(t)) \leq \psi(\eta((1 - \lambda)t)) - \varphi(\eta((1 - \lambda)t)) + Mt(\lambda - 1)^2.$$

If $m \in D^+((\psi - \varphi) \circ \eta)(0) \setminus \{0\}$, we have

$$(\mathcal{L}_t \psi)(\eta(t)) - \varphi(\eta(t)) \leq m((1 - \lambda)t) + o((1 - \lambda)t) + Mt(\lambda - 1)^2,$$

where $\lim_{\lambda \rightarrow 1} \frac{o((1-\lambda)t)}{1-\lambda} = 0$. Choosing appropriately λ close to 1, we get

$$(\mathcal{L}_t \psi)(\eta(t)) - \varphi(\eta(t)) < 0.$$

□

Proposition 18. *Suppose φ is dominated and $\psi \in \omega_{\mathcal{L}}(u)$. For any $y \in \mathcal{M}$ there exists $\gamma \in \mathcal{K}(y)$ such that the function $t \mapsto (\mathcal{L}_t \psi)(\gamma(t)) - \varphi(\gamma(t))$ is constant.*

Proof. Let $(s_k)_k$ and $(t_k)_k$ be two diverging sequences, η a curve of \mathcal{K} such that $y = \lim_k \eta(s_k)$, and ψ the uniform limit of $\mathcal{L}_{t_k} u_0$. We can assume that the curve $\gamma : \mathbb{R} \rightarrow G$, defined by $\gamma(t) = \lim_k \eta(t + s_k)$, is the local uniform limit of the sequence $\eta(s_k + \cdot)$ in \mathbb{R} , and so $\gamma \in \mathcal{K}$. We assume, in addition, that $t_k - s_k \rightarrow \infty$, as $k \rightarrow \infty$, and that $\mathcal{L}_{t_k - s_k} u_0$ converges uniformly to $\psi_1 \in \omega_{\mathcal{L}}(u_0)$. The nonexpansiveness of the Lax-Oleinik semigroup implies

$$\|\mathcal{L}_{t_k} u_0 - \mathcal{L}_{t_k} \psi_1\|_{\infty} \leq \|\mathcal{L}_{t_k - s_k} u_0 - \mathcal{L}_{s_k} \psi_1\|_{\infty}$$

which implies that $\mathcal{L}_{t_k} \psi_1$ converges uniformly to ψ . We know from Proposition 14 that the function $s \mapsto (\mathcal{L}_s \psi)(\eta(s)) - \varphi(\eta(s))$ is nonincreasing in \mathbb{R}^+ , and hence it has a limit l as $s \rightarrow \infty$, which is finite, since it is greater than or equal to $-\|u - \varphi\|_{\infty}$. Given $t > 0$, we have

$$l = \lim_{k \rightarrow \infty} (\mathcal{L}_{s_k + t} \psi_1)(\eta(s_k + t)) - \varphi(\eta(s_k + t)) = (\mathcal{L}_t \psi)(\gamma(t)) - \varphi(\gamma(t))$$

The function $t \mapsto (\mathcal{L}_t \psi)(\gamma(t)) - \varphi(\gamma(t))$ is therefore constant on \mathbb{R}^+ . From this we deduce, by applying Proposition 17 to the curve $\gamma(s + \cdot) \in \mathcal{K}$, for any fixed s , that $D^+((\psi - \varphi) \circ \gamma)(s) \setminus \{0\} = \emptyset$ for any $s \in \mathbb{R}$. This implies that $\psi - \varphi$ is constant on γ . □

Proposition 19. *Let $\eta \in \mathcal{K}$, $\psi \in \omega_{\mathcal{L}}(u)$ and v be defined by (10). For any $\varepsilon < 0$ there exists $\tau \in \mathbb{R}$ such that*

$$|\psi(\eta(\tau)) - v(\eta(\tau))| < \varepsilon.$$

Proof. Since the curve η is contained in \mathcal{A} , we have

$$v(\eta(0)) = \min_{z \in G} u_0(z) + \Phi(z, \eta(0)),$$

and hence $v(\eta(0)) = u_0(z_0) + \Phi(z_0, \eta(0))$, for some $z_0 \in G$. Take a curve $\gamma : [0, T] \rightarrow G$ such that

$$v(\eta(0)) + \frac{\varepsilon}{2} = u_0(z_0) + \Phi(z_0, \eta(0)) + \frac{\varepsilon}{2} > u_0(z_0) + \int_0^T L(\gamma, \dot{\gamma}) \geq \mathcal{L}_T u_0(\eta(0)).$$

Choosing a divergent sequence $(t_n)_n$ such that $\mathcal{L}_{t_n} u_0$ converges uniformly to ψ we have for n sufficiently large

$$\|\mathcal{L}_{t_n} u_0 - \psi\|_{\infty} < \frac{\varepsilon}{2}, \quad t_n - T > 0.$$

Take $\tau = t_n - T$

$$\begin{aligned} \psi(\eta(\tau)) - \frac{\varepsilon}{2} &< \mathcal{L}_{t_n} u_0(\eta(\tau)) = \mathcal{L}_\tau \mathcal{L}_T u_0 \\ &= \mathcal{L}_T u_0(\eta(0)) + \int_0^\tau L(\eta, \dot{\eta}) \\ \frac{\varepsilon}{2} + v(\eta(0)) + \int_0^\tau L(\eta, \dot{\eta}) &= \frac{\varepsilon}{2} + v(\eta(\tau)) \end{aligned}$$

which gives the assertion since $\psi(\eta(\tau)) - v(\eta(\tau)) \geq 0$ by Proposition 11 \square

From Propositions 18 and 19 we obtain

Theorem 6. *Let $\psi \in \omega_{\mathcal{L}}(u)$ and v be defined by (10). Then $\psi = v$ on \mathcal{M} .*

Theorem 7. *Let $u \in C(G)$, then $\mathcal{L}_t u$ converges uniformly as $t \rightarrow \infty$ to v given by (10).*

Proof. The function \underline{u} is a dominated function that coincides with v on \mathcal{M} by Theorem 6. Proposition 15 implies that \underline{u} coincide with v on \mathcal{A} and so does with w . By item (1) of Corollary 2 we have $\underline{u} \leq v$. \square

6. VISCOSITY SOLUTIONS OF THE HAMILTON - JACOBI EQUATION

In this section we compare weak KAM and viscosity solutions.

Definition 4.

- A real function φ defined on the neighborhood of v_a is C^1 if for every j with $v_a \in I_j$, $\varphi|_{I_j}$ is C^1 .
- A real function φ defined on the neighborhood of (v_a, t) is C^1 if for every j with $v_a \in I_j$, $\varphi|_{I_j \times (t - \delta, t + \delta)}$ is C^1 .

If $z : [0, \delta] \rightarrow I_j$ is differentiable and $z(0) = v_a$, then $\zeta = z'_+(0) \in T_{v_a}^- I_j$ and we have

$$D^j \varphi(v_a) \zeta = (\varphi \circ z)'_+(0).$$

We consider the Hamiltonian consisting in C^{k-1} convex super-linear functions $H_j : T^* I_j \rightarrow \mathbb{R}$ given by

$$H_j(x, p) = \max\{-pz - L_j(x, z) : z \in T_x^- I_j \text{ or } z \in T_x I_j\}$$

according to whether or not x is a vertex, and the Hamilton Jacobi equations

$$(20) \quad H(x, Du(x)) = c,$$

$$(21) \quad u_t(x, t) + H(x, D_x u(x, t)) = 0.$$

Note that if L is symmetric at the vertices, then for any vertex v_a there is a function h_a such that $H_j(v_a, p) = h_a(|p|)$ for any j with $v_a \in I_j$.

Definition 5. A function $u : G \rightarrow \mathbb{R}$ is a

- *viscosity subsolution* of (20) if satisfies the usual definition in $G - \{v_b\}$ and for any C^1 function φ on the neighborhood of v_a s.t. $u - \varphi$ has a maximum at v_a we have

$$\sup\{H_j(v_a, D^j\varphi(v_a)) : v_a \in I_j\} \leq c.$$

- *viscosity supersolution* of (20) if satisfies the usual definition in $G - \{v_b\}$ and for any C^1 function φ on the neighborhood of v_a s.t. $u - \varphi$ has a minimum at v_a we have

$$\sup\{H_j(v_a, D^j\varphi(v_a)) : v_a \in I_j\} \geq c$$

- *viscosity solution* if it is both, a subsolution and a supersolution.

A function $u : G \times [0, \infty) \rightarrow \mathbb{R}$ is a

- *viscosity subsolution* of (21) if satisfies the usual definition in $G - \{v_b\} \times [0, \infty)$ and for any C^1 function φ on the neighborhood of (v_a, t) s.t. $u - \varphi$ has a maximum at (v_a, t) we have

$$\varphi_t(v_a, t) + \sup\{H_j(v_a, D^j\varphi(v_a, t)) : v_a \in I_j\} \leq c.$$

- *viscosity supersolution* of (20) if satisfies the usual definition in $G - \{v_b\} \times [0, \infty)$ and for any C^1 function φ on the neighborhood of (v_a, t) s.t. $u - \varphi$ has a minimum at (v_a, t) we have

$$\varphi_t(v_a, t) + \sup\{H_j(v_a, D^j\varphi(v_a, t)) : v_a \in I_j\} \geq c$$

- *viscosity solution* if it is both, a subsolution and a supersolution.

Proposition 20. *If $u : G \rightarrow \mathbb{R}$ is a dominated then then it is a viscosity subsolution of (20). If u is a backward weak KAM solution then it is a viscosity solution.*

Proof. Suppose $u : G \rightarrow \mathbb{R}$ is dominated. Let φ be a C^1 function on the neighborhood of v_a s.t. $u - \varphi$ has a maximum at v_a , j s.t. $v_a \in I_j$, $z : [0, \delta] \rightarrow I_j$ differentiable with $z(0) = v_a$, $\zeta = z'(0)$. Define $\gamma : [-\delta, 0] \rightarrow I_j$ by $\gamma(s) = z(-s)$.

$$\begin{aligned} \varphi(v_a) - \varphi(\gamma(s)) &\leq u(v_a) - u(\gamma(s)) \leq \int_s^0 L_j(\gamma, \dot{\gamma}) - cs \\ \frac{\varphi(v_a) - \varphi(z(t))}{t} &\leq \frac{1}{t} \int_{-t}^0 L_j(\gamma, \dot{\gamma}) + c \\ -D^j\varphi(v_a)\zeta &\leq L_j(v_a, \zeta) + c. \end{aligned}$$

So u is a subsolution.

Let φ be a C^1 function on the neighborhood of v_a s.t. $u - \varphi$ has a minimum at v_a . Let $\gamma : (-\infty, 0] \rightarrow G$ be such that $\gamma(0) = v_a$ and for $t < 0$

$$u(v_a) - u(\gamma(t)) = \int_t^0 L_j(\gamma, \dot{\gamma}) - ct$$

Let $\delta > 0, j$ be such that $\gamma([- \delta, 0]) \subset I_j$.

$$\varphi(v_a) - \varphi(\gamma(s)) \geq \int_s^0 L_j(\gamma, \dot{\gamma}) - cs$$

Define $z : [0, \delta] \rightarrow I_j$ by $z(t) = \gamma(-t)$, $\zeta = z'(0)$,

$$\begin{aligned} \frac{\varphi(v_a) - \varphi(z(t))}{t} &\geq \frac{1}{t} \int_{-t}^0 L_j(\gamma, \dot{\gamma}) + c \\ -D^j \varphi(v_a) \zeta &\geq L_j(v_a, \zeta) + c. \end{aligned}$$

So u is a supersolution. □

Proposition 21. *Let $f : G \rightarrow \mathbb{R}$ be continuous and define $u : G \times [0, \infty) \rightarrow \mathbb{R}$ by $u(x, t) = \mathcal{L}_t f(x)$, then u is a viscosity solution of 21*

Proof. Since $\mathcal{L}_t f = \mathcal{L}_{t-s}(\mathcal{L}_s f)$ if $0 \leq s < t$, for any $\gamma : [s, t] \rightarrow G$

$$(22) \quad u(\gamma(t), t) - u(\gamma(s), s) \leq \int_s^t L(\gamma, \dot{\gamma})$$

and for any $x \in G$ there is $\gamma : [s, t] \rightarrow G$ with $\gamma(t) = x$ such that equality in (22) holds.

Let φ be a C^1 function on the neighborhood of (v_a, t) s.t. $u - \varphi$ has a maximum at (v_a, t) , j s.t. $v_a \in I_j$, $z : [0, \delta] \rightarrow I_j$ differentiable with $z(0) = v_a$, $\zeta = z'(0)$. Define $\gamma : [t - \delta, t] \rightarrow I_j$ by $\gamma(s) = z(t - s)$.

$$\begin{aligned} \varphi(v_a, t) - \varphi(\gamma(s), s) &\leq u(v_a, t) - u(\gamma(s), s) \leq \int_s^t L_j(\gamma, \dot{\gamma}) \\ \frac{\varphi(v_a, t) - \varphi(z(t - s), s)}{t - s} &\leq \frac{1}{t - s} \int_s^t L_j(\gamma, \dot{\gamma}) \\ \varphi_t(v_a, t) - D_x^j \varphi(v_a, t) \zeta &\leq L_j(v_a, \zeta). \end{aligned}$$

So u is subsolution.

Let φ be a C^1 function on the neighborhood of (v_a, t) s.t. $u - \varphi$ has a minimum at (v_a, t) . Let $\gamma : [t - 1, t] \rightarrow G$ be such that $\gamma(t) = v_a$ and

$$u(v_a, t) - u(\gamma(t - 1), t - 1) = \int_{t-1}^t L(\gamma, \dot{\gamma})$$

Let $\delta > 0, j$ be such that $\gamma([t - \delta, t]) \subset I_j$. For $s \in [t - \delta, t]$

$$\varphi(v_a, t) - \varphi(\gamma(s), s) \geq \int_s^t L_j(\gamma, \dot{\gamma})$$

Define $z : [0, \delta] \rightarrow I_j$ by $z(s) = \gamma(t-s)$, $\zeta = z'(0)$,

$$\begin{aligned} \frac{\varphi(v_a, t) - \varphi(z(t-s), s)}{t-s} &\geq \frac{1}{t-s} \int_s^t L_j(\gamma, \dot{\gamma}) \\ \varphi_t(v_a, t) - D_x^j \varphi(v_a, t) \zeta &\geq L_j(v_a, \zeta). \end{aligned}$$

So u is supersolution. \square

Proposition 22. *Suppose the Lagrangian is symmetric at the vertices. Let $u, v : G \times [0, T] \rightarrow \mathbb{R}$ be respectively a Lipschitz viscosity sub, supersolution of (21) such that $u(x, 0) \leq v(x, 0)$, for any $x \in G$. Then $u \leq v$.*

Proof. Suppose that there are x^*, t^* such that $\delta = u(x^*, t^*) - v(x^*, t^*) > 0$. Let $0 < \rho \leq \frac{\delta}{4t^*}$ and define $\Phi : G^2 \times [0, T]^2$ by

$$\Phi(x, y, t, s) = u(x, t) - v(y, s) - \frac{d(x, y)^2 + |t-s|^2}{2\varepsilon} - \rho(t+s),$$

where $d(x, y)$ is the shortest lenght of a path in G connecting x and y , and so $d(x, y) = d(y, x)$

From the previous definitions we have

$$(23) \quad \frac{\delta}{2} \leq \delta - 2\rho t^* = \Phi(x^*, x^*, t^*, t^*) \leq \sup_{G^2 \times [0, T]^2} \Phi = \Phi(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon).$$

It follows from $\Phi(x_\varepsilon, x_\varepsilon, t_\varepsilon, t_\varepsilon) + \Phi(y_\varepsilon, y_\varepsilon, s_\varepsilon, s_\varepsilon) \leq 2\Phi(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon)$ that

$$\begin{aligned} \frac{d(x_\varepsilon, y_\varepsilon)^2 + |t_\varepsilon - s_\varepsilon|^2}{2\varepsilon} &\leq u(x_\varepsilon, t_\varepsilon) - u(y_\varepsilon, s_\varepsilon) + v(x_\varepsilon, t_\varepsilon) - v(y_\varepsilon, s_\varepsilon) \\ &\leq C(d(x_\varepsilon, y_\varepsilon)^2 + |t_\varepsilon - s_\varepsilon|^2)^{1/2} \end{aligned}$$

Thus, there is a sequence $\varepsilon \rightarrow 0$ such that $x_\varepsilon, y_\varepsilon$ converge to $\bar{x} \in G$ and $t_\varepsilon, s_\varepsilon$ converge to $\bar{t} \in [0, T]$ and (23) gives

$$\frac{\delta}{2} \leq \Phi(\bar{x}, \bar{x}, \bar{t}, \bar{t}) \leq u(\bar{x}, \bar{t}) - v(\bar{x}, \bar{t}),$$

and so $\bar{t} \neq 0$. Define the test functions

$$\begin{aligned} \varphi(x, t) &= v(y_\varepsilon, s_\varepsilon) + \frac{d(x, y_\varepsilon)^2 + |t - s_\varepsilon|^2}{2\varepsilon} + \rho(t + s_\varepsilon) \\ \psi(y, s) &= u(x_\varepsilon, t_\varepsilon) - \frac{d(x_\varepsilon, y)^2 + |t_\varepsilon - s|^2}{2\varepsilon} - \rho(t_\varepsilon + s). \\ \varphi_t(x_\varepsilon, t_\varepsilon) &= \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + \rho, \quad \psi_s(y_\varepsilon, s_\varepsilon) = \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} - \rho \end{aligned}$$

Since $u - \varphi$ has maximum at $(x_\varepsilon, t_\varepsilon)$, $v - \psi$ has minimum at $(y_\varepsilon, s_\varepsilon)$, u is subsolution and v is supersolution,

$$(24) \quad \begin{aligned} 2\rho = \varphi_t(x_\varepsilon, t_\varepsilon) - \psi_s(y_\varepsilon, s_\varepsilon) \leq & \sup\{H_j(y_\varepsilon, -D_y^j\left(\frac{d(x_\varepsilon, y)^2}{2\varepsilon}\right)(y_\varepsilon)) : x \in I_j\} \\ & - \sup\{H_j(x_\varepsilon, D_x^j\left(\frac{d(x, y_\varepsilon)^2}{2\varepsilon}\right)(x_\varepsilon)) : x \in I_j\} \end{aligned}$$

Since $\rho > 0$ we can not have $x_\varepsilon = y_\varepsilon$.

Chose any orientation of the edges to write $T^*I_j = I_j \times \mathbb{R}$.

If \bar{x} is not a vertex, $\bar{x} \in I_j$, for $\varepsilon > 0$ small we have

$$D_x^j\left(\frac{d(x, y_\varepsilon)^2}{2\varepsilon}\right)(x_\varepsilon) = \pm \frac{d(x_\varepsilon, y_\varepsilon)}{\varepsilon} = -D_y^j\left(\frac{d(x_\varepsilon, y)^2}{2\varepsilon}\right)(y_\varepsilon).$$

If we denote by $a(x_\varepsilon, y_\varepsilon)$ this common value, then (24) becomes

$$2\rho \leq H_j(y_\varepsilon, a(x_\varepsilon, y_\varepsilon)) - H_i(x_\varepsilon, a(x_\varepsilon, y_\varepsilon))$$

with $a(x_\varepsilon, y_\varepsilon)$ bounded as $\varepsilon \rightarrow 0$, giving a contradiction.

Suppose now that $\bar{x} = v_a$. For $\varepsilon > 0$ small we distinguish the following cases

1. Neither x_ε nor y_ε is a vertex. If $x_\varepsilon, y_\varepsilon$ are in the same edge I_j , $d(x_\varepsilon, y_\varepsilon) = |x_\varepsilon - y_\varepsilon|$. If x_ε is in edge I_i , y_ε is in edge I_j , and $v_a \in I_i \cap I_j$, then $d(x_\varepsilon, y_\varepsilon) = d(x_\varepsilon, v_a) + d(v_a, y_\varepsilon)$. In both subcases

$$\begin{aligned} |D_x^i\left(\frac{d(x, y_\varepsilon)^2}{2\varepsilon}\right)(x_\varepsilon)| &= \frac{d(x_\varepsilon, y_\varepsilon)}{\varepsilon} \\ |D_y^j\left(\frac{d(x_\varepsilon, y)^2}{2\varepsilon}\right)(y_\varepsilon)| &= \frac{d(x_\varepsilon, y_\varepsilon)}{\varepsilon} \end{aligned}$$

Then (24) becomes

$$2\rho \leq H_j(y_\varepsilon, \pm \frac{d(x_\varepsilon, y_\varepsilon)}{\varepsilon}) - H_i(x_\varepsilon, \pm \frac{d(x_\varepsilon, y_\varepsilon)}{\varepsilon}).$$

2. Suppose $x_\varepsilon = v_a$, $y_\varepsilon \in I_j \setminus \{v_a\}$.

$$\begin{aligned} |D_x^j\left(\frac{d(x, y_\varepsilon)^2}{2\varepsilon}\right)(v_a)| &= \pm \frac{d(v_a, y_\varepsilon)}{\varepsilon} \\ |D_y^j\left(\frac{d(v_a, y)^2}{2\varepsilon}\right)(y_\varepsilon)| &= \pm \frac{d(v_a, y_\varepsilon)}{\varepsilon} \end{aligned}$$

Since

$$H_j(v_a, \pm \frac{d(v_a, y_\varepsilon)}{\varepsilon}) = h_a(v_a, \frac{d(v_a, y_\varepsilon)}{\varepsilon}),$$

we have that (24) becomes

$$2\rho \leq H_j(y_\varepsilon, \pm \frac{d(x_\varepsilon, y_\varepsilon)}{\varepsilon}) - H_j(x_\varepsilon, \frac{d(x_\varepsilon, y_\varepsilon)}{\varepsilon}).$$

3. If $y_\varepsilon = v_a$, $x_\varepsilon \in I_j \setminus \{v_a\}$ we get in the same way that (24) becomes

$$2\rho \leq H_j\left(y_\varepsilon, \frac{d(x_\varepsilon, y_\varepsilon)}{\varepsilon}\right) - H_j\left(x_\varepsilon, \pm \frac{d(x_\varepsilon, y_\varepsilon)}{\varepsilon}\right).$$

Since $\frac{d(x_\varepsilon, y_\varepsilon)}{\varepsilon}$ remains bounded as $\varepsilon \rightarrow 0$, in all cases we get a contradiction. \square

Corollary 6. *Suppose the Lagrangian is symmetric at the vertices. Let $u, v : G \times [0, T] \rightarrow \mathbb{R}$ be viscosity solutions of (21) such that $u(x, 0) = v(x, 0)$ for any $x \in G$. Then $u = v$.*

Corollary 7. *Suppose the Lagrangian is symmetric at the vertices. Let $f : G \rightarrow \mathbb{R}$ be a viscosity solution of (20), then f is a fixed point of the Lax semigroup $\mathcal{L}_t + ct$.*

Proof. We will show that $u(x, t) = f(x) - ct$ is a viscosity solution of (21). Proposition 21 and Corollary 6 then imply that $f - ct = \mathcal{L}_t f$.

Let φ be a C^1 function on the neighborhood of (v_a, t) s.t. $u - \varphi$ has a maximum at (v_a, t) . Then $s \rightarrow -cs - \varphi(v_a, s)$ has a maximum at t and so $\varphi_t(v_a, t) = -c$. Since $f - \varphi(\cdot, t)$ has a maximum at v_a we have

$$\sup\{H_j(x, D^j\varphi(x)) : x \in I_j\} \leq c = -\varphi_t(v_a, t),$$

so u is a subsolution of (21). Similarly u is a supersolution of (21). \square

Corollary 8. *Suppose the Lagrangian is symmetric at the vertices. Let $u : G \rightarrow \mathbb{R}$ be a viscosity solution of (20) then formula (9) holds.*

Proof. By Proposition 20 and Corollary 7, u is a backward weak KAM solution and by Theorem (3), formula (9) holds. \square

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